Asymptotic behavior of sequences

UIP exercise lesson

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A document from 1648

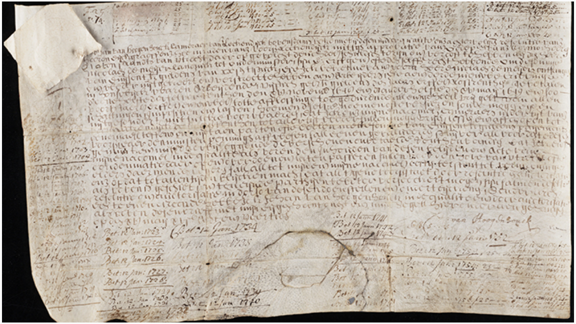
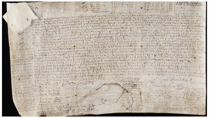


Figure 1: Yale's Beinecke Rare Book & Manuscript Library

This is a bond from the Dutch water board Stichtse Rijnlanden, issued on May 15th 1648.

In exchange of 1000 "Carolus guilder" paid at the time, the bearer of the document receives 20 guilder every year of interest.

A document from 1648



- The bond is still valid today and Yale in 2010-2015 received ∼10E/year from it.
- It is hard to estimate the purchasing power of 1000 guilders in the 17th century, but as a reference: a pastor earned 500 guilder per year, roughly 20 guilders every two weeks.
- Interestingly, the bond never expires, but is perpetually valid.

An investment opportunity?

Perpetual bonds

Perpetual bonds do no expire, they cannot be redeemed, but they only pay the owner an annual interest forever.

You are offered to buy such a perpetual bond. You pay 1,000,000kr today, and you will receive 20,000kr every year for the rest of eternity.

Questions:

1. Is is a good deal? Would you accept it?
2. Will the bank go bankrupt? Will you be infinitely rich?

Inflation

Things get nominally more expensive as time passes.

Example

Let us assume for simplicity that there is an annual inflation rate of 2%.

This means that what we can buy today for 20,000kr, it will cost 20,400kr next year, 20,808kr in two year, 24,380kr in 10 years.

The interest that we will receive from the bond will be always the same number (20,000kr), but it will be worth less and less relative to the cost of life.

Inflation

So in today money, our bond interest will be worth

1. $(1-2\%) \times 20{,}000$kr $= 19{,}600$kr the first year;
2. $(1-2\%) \times 19{,}600$kr $= 19{,}208$kr the second year;
3. $(1-2\%) \times 19{,}208$kr $= 18{,}824$kr the third year...

and so on.

So the total gain we obtain from our potential bond is

1. $M_1 = 19,600$kr the first year;
2. $M_2 = 19,600 + 19,208 = 38,808$kr the second year;
3. $M_3 = 38,808 + 18,824 = 57,632$kr the third year...

Handling infinities

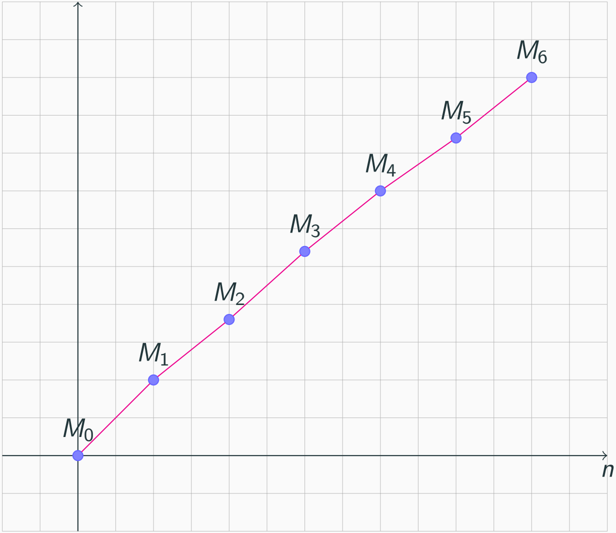
We can calculate "easily" M_Y for any given year Y.

But to to properly estimate the predicted income from the bond (and thus, how much we would be available to pay), we need to know the total amount gained over "infinity" years.

Problem

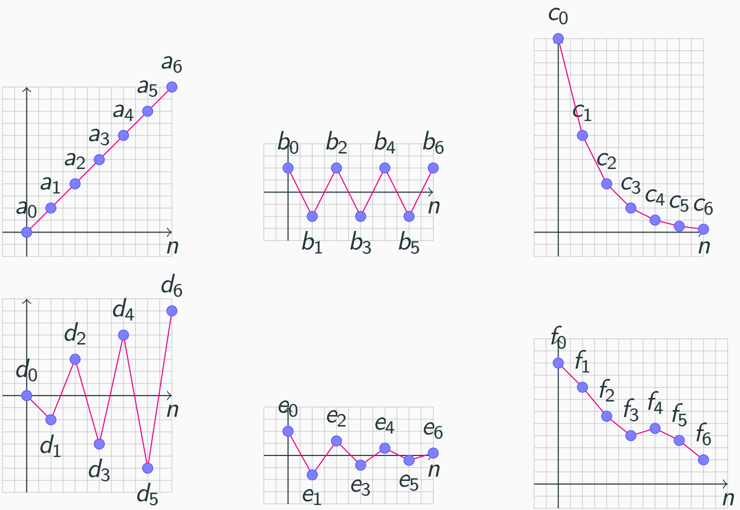
How can we describe (mathematically) this "value after infinite time" of our investment?

A sequence of numbers



Exercise

Divide these cases into two groups which share some common property.



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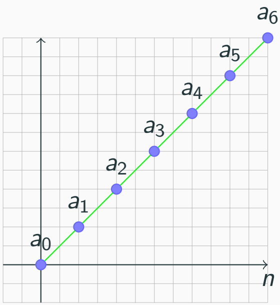

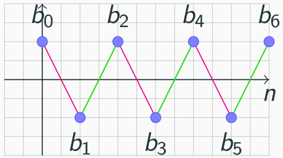

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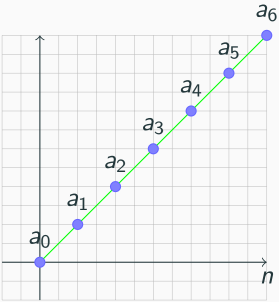
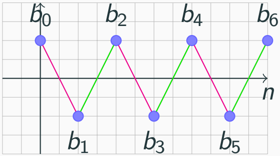


(a) Monotone



(b) Non monotone

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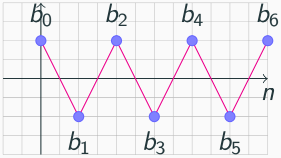
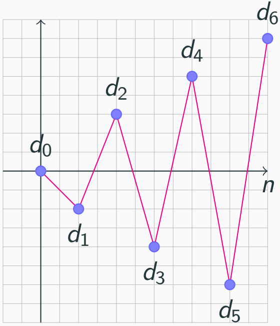
Exercise





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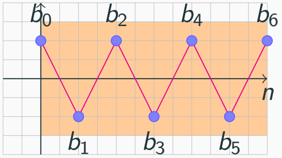
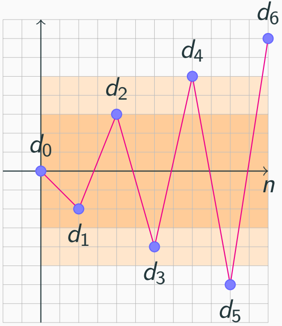


(a) Bounded



(b) Unbounded

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Back to investments

1. If the gains from our investment M_Y are **monotone**, we will get more money every year.
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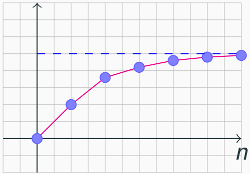
1. If the gains from our investment M_Y are **monotone**, we will get more money every year.
2. Otherwise, we risk loosing money (for example with company shares).
3. If the gains M_Y are **monotone** and **unbounded**, the bank will bankrupt and we will be infinitely rich (not really realistic).

Monotone and bounded

What if M_Y is **monotone** and **bounded**?

1. Can it oscillate between two values?



Monotone and bounded

What if M_Y is **monotone** and **bounded**?

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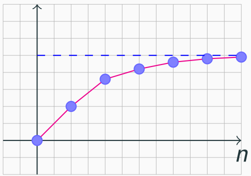

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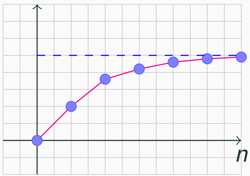
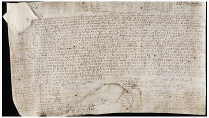


M_Y will get closer and closer to some maximum value which is called the limit.
The expected gains are finite!

Perpetual bonds



In the case of perpetual bond, it is possible to actually calculate the total gain possible (the limit):

$$M_\infty = \frac{\text{annual interest}}{\text{annual inflation rate}}.$$

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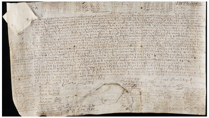

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So that in our case

$M_\infty = 20{,}000\text{kr}/0.02 = 1{,}000{,}000\text{kr}$ (we break even)!

ILOs

After the lecture, the students should be able to:

1. classify sequences according to the properties of *monotonicity* and *boundedness*;
2. argue that monotone bounded sequences have a well-defined *limit*;
3. explain the value of perpetual bonds via asymptotic analysis of predicted gains.