Lecture 1 - 21/11/22

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- Elementary objects : Spins and qudits

$\mathcal{H} = \mathbb{C}^d$ = qudit Finite dim Hilbert space

or spin $\frac{d-1}{2}$ particle

Dirac notation:

elements of $\mathbb{C}^d$ are "ket"s

$|\varphi\rangle \in \mathbb{C}^d$. Scalar product $\langle \varphi_1 | \varphi_2 \rangle$ (antilinear, linear)

elements of $(\mathbb{C}^d)^*$ are "bra"s

$\langle \psi | \in \mathbb{C}^d$

is the Linear functional given by

$|\varphi\rangle \longmapsto \langle \psi | \varphi \rangle$

we always assume there is a preferred orthonormal basis of $\mathbb{C}^d$

$\{ |e_0\rangle, \dots, |e_{d-1}\rangle \}$

dual basis $\{ \langle e_0|, \dots, \langle e_i| \}$

So if $|\varphi\rangle = \sum_{i=0}^{d} \varphi_i |e_i\rangle$ "colum vector"

then $\langle \varphi | = \sum_{i=0}^{d-1} \bar{\varphi}_i \langle e_i |$ "row vector"

Rank-one maps: $L = |\varphi\rangle\langle\psi|$

$L|\chi\rangle = \langle \psi | \chi \rangle \cdot |\varphi\rangle$

Adjoints: $L^* = |\psi\rangle\langle\varphi|$

- Operators, States, measures

Hermitian Operators on $\mathcal{H}$ represent "physical properties". → "Observables"

eg. if $\mathcal{H} = \mathbb{C}^2$,

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$= |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$$

magnetization

Quantization: the values a certain property can take are discrete: they are given by the spectrum of the operator.

if M is an Hermitian operator with eigenvalue λ and (normalized) eigenvector $|\varphi\rangle$

$$\langle \varphi | M \varphi \rangle = \lambda$$

"interpretation": the value of property M at $|\varphi\rangle$ is λ.
if $|\varphi\rangle$ is not an eigenvector, then we will see that the value of property M is "not deterministic".

A state $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is a linear, positive, normalized functional.

<u>Positive</u> $\omega(X^* X) \geq 0$

$M \in \mathcal{B}(\mathcal{H})$ is positive semi definite iff. $M = X^* X$ for some $X \in \mathcal{B}(\mathcal{H})$

i.e. ω is positive is equivalent to saying that it maps positive elements to positive elements

<u>normalized</u> $\omega(\mathbb{1}) = 1$.

<u>OBS</u>: the set of states is convex: if ω_0 and ω_1 are states, then

$\cdot\ \omega_p = (1-p)\,\omega_0 + p\,\omega_1$ $\quad p \in [0,1]$

is a state.

<u>Def</u>: ω is PURE if it is an extremal point of the set of states, i.e. if it cannot be written as a non-trivial convex combination of states.

<u>Claim (see exercise sheet)</u>

ω is pure iff $\omega(X) = \langle \varphi | X \varphi \rangle$ for some normalized $|\varphi\rangle \in \mathcal{H}$.

Density operator / density matrix

Given a state ω, there exists a $\rho \in B(\mathcal{H})$

positive semi definite, $\operatorname{Tr} \rho = 1$, such that

$$\omega(X) = \operatorname{Tr}(\rho X) \qquad \forall X \in B(\mathcal{H})$$

ρ is called the density matrix of ω.

extremal points of density matrices = rank-one projections.

Spectral thm $\quad \rho = \sum_{i=0}^{d-1} p_i \, |\varphi_i\rangle\langle\varphi_i|$

$(p_i)_{i=0}^{d-1}$ prob. distr. $\qquad p_i \geq 0 \quad \sum_i p_i = 1$

if $\rho = |\varphi\rangle\langle\varphi|$ then $\operatorname{Tr}(\rho X) = \langle\varphi | X\varphi\rangle$.

<u>example</u>: maximally mixed state

$$X \mapsto \frac{\operatorname{Tr}(X)}{d}$$

density matrix $\quad \frac{\mathbb{1}}{d}$

Can be seen as the uniform mixture of any basis of pure states

$$\frac{\mathbb{1}}{d} = \sum_{i=0}^{d-1} \frac{1}{d} |e_i\rangle\langle e_i|$$

Terminology: $\omega(X)$ is the expectation value of X w.r.t. state ω.

Why?, we can think of $\omega(x) = \sum_i p_i \langle \varphi_i | x \varphi_i \rangle$

as a probabilistic mixture of pure states.

ensemble of pure states $\{ (|\varphi_i\rangle, p_i) \}_i$

state $|\varphi_i\rangle$ with prob p_i.

Measurement:

A Positive operator-valued measure (POVM)

is a family of positive semidefinite operators $\{F_i\}_i$

such that
$$\sum_i F_i = \mathbb{1}$$

In other words, they are a partition of the identity.

If $\{F_i\}_i$ is a set of orthogonal projections
$$F_i F_j = \delta_{ij} F_i$$

Then we call this a Projection-valued measure.

<u>OBS</u> each observable = Hermitian operator defines a PVM via the spectral THM:
$$M = \sum_i \lambda_i P_i \qquad \{P_i\}_i \text{ is a PVM}$$

Naimark dilation theorem

If $\{F_i\}_i \subseteq B(\mathcal{H})$ is a POVM, then there exist a

isometry $V: \mathcal{H} \to K$ $\quad$ K Hilbert space

and a PVM $\{P_i\}_i$ on K such that
$$F_i = V^* P_i V$$

<u>Proof:</u> Let $K = \mathcal{H} \otimes \mathcal{H}$, $\quad P_i = \mathbb{1} \otimes |i\rangle\langle i|$,

$$V = \sum_i \sqrt{F_i} \otimes |i\rangle$$

Measurement axioms

In a quantum mechanical experiment, when our system is in state ω, we can "perform a measurement" described by a POVM $\{F_i\}_i$

The outcome of the measurement is one of the labels i, with prob.

$$\mathrm{Prob}(i) = \omega(F_i)$$

! If our measurement is a PVM, and we obtain outcome i, then after the measurement the system will be in state

$$\omega_i(X) := \frac{\omega(F_i X F_i)}{\omega(F_i)}$$

("post-measurement state")

OBS if $M = \sum_i \lambda_i P_i$ is an observable and $|\psi_{i_0}\rangle$ is eigenstate with eigenvalue λ_{i_0} then when we measure M on state $|\psi_{i_0}\rangle$ we get

$$\mathrm{Prob}(i) = \langle \psi_{i_0} | P_i \psi_{i_0}\rangle = \delta_{i,i_0}$$

and after measuring we are in the same state $|\psi_{i_0}\rangle$,

deterministic!

If instead we are in state

$$|\psi\rangle = \sum_i \mu_i |\psi_i\rangle \quad \text{with}$$

$|\psi_i\rangle$ eigenstate of M with eigenvalue λ_i

Then $\quad \text{Prob}(i) = |\mu_i|^2$

This undeterministic outcome is a consequence of the choice of the measurement: $|\psi\rangle$ is pure, there is no "uncertainty" about its state, e.g. there is one PVM which perfectly distinguish the situation

$$P_0 = |\psi\rangle\langle\psi| \qquad P_1 = \mathbb{1} - |\psi\rangle\langle\psi|$$

On the other hand, mixed states are "classically" random.