Lezione 2 - 23/11/22

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COMPOSITE SYSTEMS

If $\mathcal{H}_A$ and $\mathcal{H}_B$ represent two independent quantum systems, what happens when we "put them together"? Which Hilbert space describes the joint system A+B?

Tensor Product: the joint system is represented by

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

If $\{|a_0\rangle, \dots, |a_{n-1}\rangle\}$ is a o.n.b. of $\mathcal{H}_A$ and $\{|b_0\rangle, \dots, |b_{m-1}\rangle\}$ is a o.n.b. of $\mathcal{H}_B$ then

$\left\{ |a_i\rangle \otimes |b_j\rangle \;\middle|\; \begin{matrix} i = 0, \dots, n-1 \\ j = 0, \dots, m-1 \end{matrix} \right\}$ is a o.n.b. of $\mathcal{H}_A \otimes \mathcal{H}_B$

Dirac's notation: $|a_i\, b_j\rangle$ and $|a_i\rangle|b_j\rangle$ are short-hands for $|a_i\rangle \otimes |b_j\rangle$

Scalar product on $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\langle a_1| \otimes \langle b_1| \cdot |a_2\rangle \otimes |b_2\rangle$$

$$= \langle a_1|a_2\rangle \cdot \langle b_1|b_2\rangle$$

and extend by linearity.

Schmidt decomposition

Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Then there exists o.n.b.

$\{|a_0\rangle, \dots, |a_{n-1}\rangle\}$ of $\mathcal{H}_A$ and

$\{|b_0\rangle, \dots, |b_{n-1}\rangle\}$ of $\mathcal{H}_B$ such that

$$|\psi\rangle = \sum_i \lambda_i \, |a_i\rangle \otimes |b_i\rangle$$

! There is only one summation index!

Proof: exercise sheet.

Def: if $|\psi\rangle = |u\rangle \otimes |v\rangle$ we say it is a product state

Otherwise, we say it is entangled.

For mixed states, we say that ω is separable if it is a convex combination of product (mixed) states

$$\omega = \sum_i p_i \; \omega_A^i \otimes \omega_B^i$$

Otherwise ω is entangled

(Obs: a pure state $|\psi\rangle$ is separable $\Leftrightarrow$ it is product so the two definitions of entanglement are consistent)

Example Bell states

The following is a o.n.b. of $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle \pm |11\rangle \right)$$

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle \pm |10\rangle \right)$$

Terminology: $|\phi^+\rangle$ is called a EPR pair

$|\psi^-\rangle$ is called a singlet state

The Schmidt decomposition gives us a way to quantify how entangled a (pure) state is

Entanglement entropy

$$|\psi\rangle = \sum_i \lambda_i \, |a_i\rangle |b_i\rangle$$

$$S = -\sum_i |\lambda_i|^2 \log |\lambda_i|^2$$

$$S \in [0, \log \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)]$$

All Bell states have the maximal e.e. for a state in $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\log 2$.

States with this property are called maximally entangled and they are of the form

$$\frac{1}{\sqrt{d}} \sum_{i=1}^{d} |u_i\rangle |v_i\rangle$$

Reduced density matrix

Let $\omega : B(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathbb{C}$ a state with reduced density matrix $\rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$

There is a natural embedding

$$\begin{aligned} B(\mathcal{H}_A) &\hookrightarrow B(\mathcal{H}_A \otimes \mathcal{H}_B) \\ X &\longmapsto X \otimes \mathbb{1}_B \end{aligned}$$

and similarly for $B(\mathcal{H}_B)$

So we can define "marginals"

$$\omega_A(X) = \omega(X \otimes \mathbb{1}_B) \qquad X \in B(\mathcal{H}_A)$$
$$\omega_B(X) = \omega(\mathbb{1}_A \otimes \ ) \qquad X \in B(\mathcal{H}_B)$$

What are the corresponding density matrices?

Answer: partial traces.

$\exists$ unique linear

$$\begin{tikzcd} & B(\mathcal{H}_A \otimes \mathcal{H}_B) \arrow[dl, "\mathrm{Tr}_B"'] \arrow[dr, "\mathrm{Tr}_A"] & \\ B(\mathcal{H}_A) \arrow[dr, "\mathrm{Tr}"'] & & B(\mathcal{H}_B) \arrow[dl, "\mathrm{Tr}"] \\ & \mathbb{C} & \end{tikzcd}$$

s.t. $\mathrm{Tr}_A(X \otimes Y) = \mathrm{Tr}(X)\, Y$

$\mathrm{Tr}_B(X \otimes Y) = \mathrm{Tr}(Y)\, X$

<u>CLAIM</u> density matrix of ω_A is $\mathrm{Tr}_B\, \rho$

$$\omega_A(x) = \mathrm{Tr}\left((x \otimes \mathbb{1}_B)\, \rho \right) = \mathrm{Tr}\left(x \cdot \mathrm{Tr}_B(\rho) \right)$$

Going back to the schmidt decomposition:

if $|\varphi\rangle = \sum_i \lambda_i\, |a_i\rangle |b_i\rangle$

then $\rho = |\varphi\rangle\langle\varphi| = \sum_{i,j} \lambda_i \lambda_j\, |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$

and $\mathrm{Tr}_A\, \rho = \sum_i |\lambda_i|^2\, |b_i\rangle\langle b_i|$

$\mathrm{Tr}_B\, \rho = \sum_i |\lambda_i|^2\, |a_i\rangle\langle a_i|$

Consequences

- they have the same spectrum!

  $\{ |\lambda_i|^2 \}_i$

- entanglement entropy is the von Neumann entropy of either

  $S(\rho) = -\mathrm{Tr}\, \rho \log \rho$

⚠ Entropy of mixed states $\neq$ entanglement

$S(|\phi^\pm\rangle\langle\phi^\pm|) = 0$

but $S(\mathrm{Tr}_A(|\phi^\pm\rangle\langle\phi^\pm|)) = \log 2$

On the other hand

$S\left(\frac{\mathbb{1}}{2} \otimes \frac{\mathbb{1}}{2}\right) = \log 4$ but this is a

product state!

Quantum mutual information

$$I := S(\mathrm{Tr}_A \rho) + S(\mathrm{Tr}_B \rho) - S(\rho)$$

For pure states = 2 Entanglement entropy

but for product states $I = 0$

Since $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$

Hamiltonians and evolutions

A special observable H will play an important role: it will describe the energy in the system and we will call it $\boxed{\text{Hamiltonian}}$

It comes with some "specialized" jargon

eigen value $\longleftrightarrow$ energy level

Smallest eigenvalue E_0 $\longleftrightarrow$ ground state energy

eigen vector with eigenvalue E_0 $\longleftrightarrow$ ground state

examples on $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$H_{\text{ISING}} = \sigma^z \otimes \sigma^z = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

energy $+1$ to $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$

energy -1 to $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$

The energy is the infinitesimal generator of time evolution

Schrödinger equation

$$\frac{d}{dt} |\psi_t\rangle = -i H |\psi_t\rangle$$

...

Solution $|\psi_t\rangle = e^{-itH} |\psi_0\rangle$

$U_t = e^{-itH}$ is a Semigroup of unitary operators

Heisenberg equation

States are constants, observables evolve!

$$\frac{d}{dt} A_t = i [H, A_t]$$

$[X, Y] := XY - YX$ commutant.

Solution $A_t = e^{itH} A_0 e^{-itH} = U_t^* A_0 U_t$

Relation between the two:

$$\langle \psi_t | A_0 \psi_t \rangle = \langle \psi_0 | A_t \psi_0 \rangle$$