Lezione 3 - 28/11/2022

Friday, 25 November 2022 16:13

Many-body quantum systems

Why we want to study infinite systems of particles?

Some phenomena only happens in that setting.

example: phase transitions.

Thermal equilibrium state $\omega_\beta(x) = \frac{Tr(e^{-\beta H}x)}{Z_\beta}$

If $H(\mu)$ depends smoothly on a parameter μ

then ω_β^μ is also smooth $\rightarrow$ phase transitions are non-smooth!

(classical) magnetization in Ising model

How can we study infinite systems?

Let Γ be a graph equipped with graph distance d

(more generally metric space)

examples $\Gamma = \mathbb{Z}$; $\Gamma = \mathbb{Z} \times \mathbb{Z}$; $\Gamma =$ hexagonal lattice

at each site $x \in \Gamma$, we associate a finite dim quantum system $\mathcal{H}_x \simeq \mathbb{C}^{d_x}$

d_x can depend on x but for simplicity I will assume that $\sup_x d_x < +\infty$

<u>Question</u>: how to make sense of $\bigotimes_{x\in\Gamma} \mathcal{H}_x$?

Naive attempt at defining a tensor product:

$$a = (|a_x\rangle)_{x\in\Gamma} \qquad |a_x\rangle \in \mathcal{H}_x$$

$$b = (|b_x\rangle)_{x\in\Gamma} \qquad |b_x\rangle \in \mathcal{H}_x$$

$$\langle b, a\rangle \overset{?}{=} \prod_{x\in\Gamma} \langle b_x | a_x\rangle$$

$\Downarrow$

It might not converge! e.g. $|b_x\rangle = (-1)^{\delta(x,0)} |a_x\rangle$

There are ways to solve this issue, but instead we will take a different route.

Local observables

Let $\Lambda \subset\subset \Gamma$ finite. Define

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x \qquad \mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda) = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x)$$

If $\Lambda_1 \subseteq \Lambda_2$, then there is a natural inclusion

$$\mathcal{A}_{\Lambda_1} \hookrightarrow \mathcal{A}_{\Lambda_2}$$

$$X \longmapsto X \otimes \mathbb{1}_{\Lambda_2}$$

Now let $(\Lambda_n)_n$ an increasing $(\Lambda_{n+1} \supseteq \Lambda_n)$ and absorbing $(\forall x\in\Gamma\ \exists n$ such that $x\in\Lambda_n)$ sequence.

<u>examples</u> $\Lambda_n = [-n, n] \subseteq \mathbb{Z}$ or $\Lambda_n = [-n,n]^{\times 2} \subseteq \mathbb{Z}^2$

Then $\mathcal{A}_{loc} := \bigcup_n \mathcal{A}_n$ local observables (operators with finite support)

and $\mathcal{A}_\Gamma = \overline{\mathcal{A}_{loc}}^{\|\cdot\|}$ ⟶ operator norm (largest s.v.)

$\mathcal{A}_\Gamma$ is a unital C^*-algebra. Each $\mathcal{A}_\Lambda$ is a $*$-subalgebra of $\mathcal{A}_\Gamma$.

def: support

$M \in \mathcal{A}_{loc}$ $\quad \operatorname{supp} M$ = smallest Λ s.t. $M \in \mathcal{A}_\Lambda$

i.e. M acts as $\mathbb{1}$ outside $\operatorname{supp} M$

Phylosophy; Study $\mathcal{A}_\Gamma$ through the finite volumes Λ.

Idea: take Λ Large but finite. Construct objects which either

- do not depend explicitely on $|\Lambda|$, or
- we can control their behavior as $|\Lambda| \to +\infty$

Obtain a corresponding object on $\mathcal{A}_\Gamma$.

Infinite volume dynamics: Local Hamiltonians

Local interaction: $X \subset\subset \Gamma$ finite $\longmapsto h_X \in \mathcal{A}_X$ Hermitian.

if $h_X = 0$ whenever $\operatorname{diam}(X) > r_0$
Then we say it has range r_0

} we will mostly focus on this case

exponential decay : $\|h_X\| \leq c \cdot e^{-\mu \operatorname{diam}(X)}$

Usually exp. decay behaves similarly to finite range.

Polynomial decay on the other hand is quite different.

example: $\Gamma = \mathbb{Z}$

Ising model $h_{ij} = \sigma_i^z \otimes \sigma_j^z$ if $|i-j| = 1$ and 0 otherwise.

Cluster state $h_v = \sigma_{v-1}^z \otimes \sigma_v^x \otimes \sigma_{v+1}^z$

Local Hamiltonian. $\Lambda \subset\subset \Gamma$ finite

"open b.c."

$$H_\Lambda := \sum_{X \subseteq \Lambda} h_X$$

H_Λ is Hermitian and $\alpha_t^\Lambda(X) = e^{itH_\Lambda} X e^{-itH_\Lambda}$

Obs $\|H_\Lambda\|$ can grow with $|\Lambda|$

$\Rightarrow \lim_{\Lambda \nearrow \Gamma} H_\Lambda = ???$ most probably not in $\mathcal{A}_\Gamma$

α_t^Λ is unital $*$-homomorphism, $\|\alpha_t^\Lambda\| = 1$

$\Rightarrow$ Can we make sense of $\lim_{\Lambda \nearrow \Gamma} \alpha_t^\Lambda$?

OBS $\alpha_t^\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$, but we can extend it to $\mathcal{A}_\Gamma$.

if $X \otimes y \in A_{loc}$ with $x \in A_\Lambda$ then

$$\alpha^\Lambda_t(x) = \alpha^\Lambda_t(x) \otimes y$$

So by linearity $\alpha^\Lambda_t : A_{loc} \to A_{loc}$.

Let $(X_n)_n \subseteq A_{loc}$ such that $X_n \to X \in A_\Gamma$ i.e.

$$\|X_n - X\| \to 0 \quad \text{as } n \to +\infty$$

$$\|\alpha^\Lambda_t(X_n) - \alpha^\Lambda_t(X_m)\| = \|\alpha^\Lambda_t\| \cdot \|X_n - X_m\|$$

$$\to 0 \quad \text{as } n \to +\infty$$

$\Rightarrow$ we can extend $\alpha^\Lambda_t : A_\Gamma \to A_\Gamma$

<u>def:</u> STRONG CONVERGENCE

$\alpha_n : A \to A$ $\quad \alpha_n \to \alpha$ strongly if

$$\|\alpha_n(M) - \alpha(M)\| \xrightarrow{n \to +\infty} 0 \quad \forall M \in A$$

we want to find conditions which allow us to show that

$$\alpha^\Lambda_t \to \alpha^\Gamma_t$$

this will require us to show that $\quad \forall t, \forall x.$

$$\|\alpha^{\Lambda_n}_t(x) - \alpha^{\Lambda_m}_t(x)\| \to 0 \quad \text{as } n,m \to \infty$$

<u>Important</u>: we **FIX** X and t, **THEN** we let $\Lambda \nearrow \Gamma$!

Lieb-Robinson's bounds

<u>Idea:</u> speed of sound in a material

1st order approx

$$\hat{\alpha}_t(M_X) = X + t\,[H_\Lambda, M_X] + o(t)$$

$$= X + t \sum_{Y \subseteq \Lambda} [h_Y, M_X] + o(t)$$

$$= X + t \underbrace{\sum_{\substack{Y \subseteq \Lambda \\ Y \cap X \neq \emptyset}} [h_Y, M_X]}_{(*)} + o(t)$$

If H is finite range,

then $(*)$ does not depend on Λ but only on

$$X(1) = \{ x \in \Lambda \mid \operatorname{dist}(x, X) \leq 1 \}$$
$$= X + B_0(1)$$

2nd order

$$\alpha^\Lambda_t(M_X) = X + t \sum_{\substack{Y \subseteq \Lambda \\ Y \cap X \neq \emptyset}} [h_Y, M_X]$$

$$+ \frac{t^2}{2} \sum_{\substack{Y_1 \subseteq \Lambda \\ Y_1 \cap X \neq \emptyset}} \sum_{\substack{Y_2 \subseteq \Lambda \\ Y_2 \cap X(2) \neq \emptyset}} [Y_2, [Y_1, X]] + o(t^2)$$

$\Rightarrow$ We need to go to order t^d $\quad d = \operatorname{dist}(X, \Lambda^c)$

to see the dependence on Λ.

If we fix t and X and let $\Lambda \nearrow \Gamma$, then $\alpha^\Lambda_t[X]$ is Cauchy!

THM (Lieb-Robinson 1972)

Fix Γ and a local interaction $\{h_X\}_{X \subset \Gamma}$. Then there exist positive constants C, μ, v such that:

For any $\Lambda \subset\subset \Gamma$ finite

Let A supported on X, B supported on Y

$X, Y \subseteq \Lambda$ $\quad d = \mathrm{dist}(X, Y) > 0$

$A(t) = \alpha_t^\Lambda(A)$

$$\| [A(t), B] \| \leq C \cdot \|A\| \cdot \|B\| \cdot e^{\mu(vt - d)}$$

v is called Lieb-Robinson or group velocity

<u>Obs 1</u>: R.h.s. is small if $d \gg vt$

$\Rightarrow$ v is a "velocity" $\sim \frac{\text{distance}}{\text{time}}$.

<u>OBS 2</u>: If A is such that

$$\| [A, B] \| \leq \varepsilon \|A\| \cdot \|B\|$$

for all B supported on Y

then $\exists A'$ supported on Y^c s.t.

$$\| A - A' \otimes \mathbb{1}_Y \| \leq \varepsilon \|A\|$$

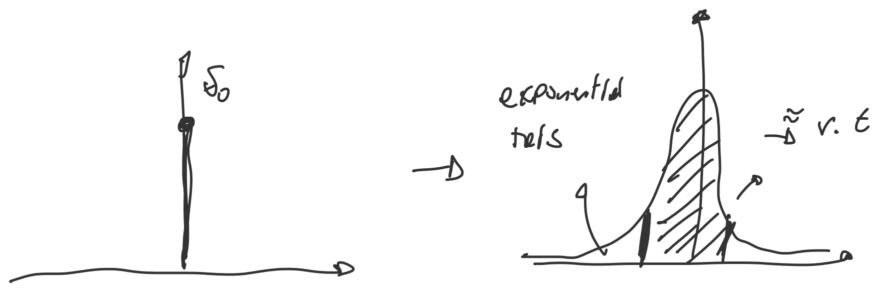
<u>Proof</u>: exercise

$\Rightarrow$ LR bounds imply that, if $t \ll \frac{d}{v}$, then

$\alpha_t^\Lambda(A)$ has support "almost" in $X(d)$,

up to exponential tails.

Similar to what happens in heat equation



$t = 0$ $\quad\quad$ $t = \varepsilon$

Cor Let $\Lambda \subset\subset \Gamma$ and $X \subseteq \Lambda$.

Let $d = \mathrm{dist}(X, \Lambda^c)$ and A supported on X

Consider $A(t) = \alpha^\Lambda_t(A)$ for $0 \leq t \leq t_{max} \ll \frac{d}{v}$

in the sense that $C e^{\mu(v t_{max} - d)} \leq \epsilon$

For some fixed $\epsilon > 0$.

Then $A(t)$ is well approximated by $\tilde{A}(t)$

$$A(t) = \alpha_t^{X(c.v.t)}(A)$$

$$\| A(t) - \tilde{A}(t) \| \leq \epsilon \|A\| \quad \forall \; 0 \leq t \leq t_{max}$$

And by construction $\mathrm{supp}\, \tilde{A}(t) = X(c.v.t)$