Lecture 4 - 30/11/2022

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Proof of Lieb-Robinson bounds

$$\| [A(t), B] \| \leq c \, \|A\| \cdot \|B\| \cdot |\mathrm{supp} A| \cdot |\mathrm{supp} B| \, e^{\mu(vt-d)} \quad (*)$$

Proof:

Let

$$C_A(Y; t) = \sup_{B \in \mathcal{A}_Y} \frac{\| [A(t), B] \|}{2 \|B\|}$$

we need to bound $C_A(y;t)$ as in $(*)/\|B\|$

Trivially $C_A(y;0) \leq \begin{cases} \|A\| & \text{if } X \cap \mathrm{supp} B \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$

Now consider

$$f(t) = [A(t), B] \qquad f: \mathbb{R} \to \mathcal{A}_\Lambda$$

$$f(0) = 0$$

$$\frac{d}{dt} f(t) = \left[\frac{d}{dt} A(t), B \right] = (*)$$

Remember $A(t) = \alpha^\Lambda_t (A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$

$$= \exp(i t \delta_{H_\Lambda})(A)$$

with $\delta_{H_\Lambda}(A) = [H_\Lambda, A]$

therefore $\frac{d}{dt} A(t) = i \delta_{H_\Lambda} [A(t)] = i [H_\Lambda, A(t)]$

$$(*) = i [[H_\Lambda, A(t)], B] = i \sum_{z \subseteq \Lambda} [[h_z, A(t)], B]$$

$$= -i \sum_{z \subseteq \Lambda} \delta_B \circ \delta_{h_z} (A(t))$$

$$\delta_B \circ \delta_{h_z} = [B, [h_z, \cdot]] .$$

Now if $[B, h_z] = 0$ then

$$\delta_B \circ \delta_{h_Z}(A) = [B, h_Z A - A h_Z]$$

$$= \underbrace{B h_Z A}_{1} - \underbrace{B A h_Z}_{2} - \underbrace{h_Z A B}_{3} + \underbrace{A h_Z B}_{4}$$

$$\delta_{h_Z} \circ \delta_B(A) = [h_Z, BA - AB]$$

$$= \underbrace{h_Z B A}_{4} - \underbrace{h_Z A B}_{3} - \underbrace{B A h_Z}_{2} + \underbrace{A B h_Z}_{4}$$

$$(\delta_B \circ \delta_{h_Z} - \delta_{h_Z} \circ \delta_B)(A) = [B, h_Z] A - A [B, h_Z]$$

$$= [[B, h_Z], A]$$

(Jacobi's identity)

<u>in Conclusion</u> $[B, h_Z] = 0 \Rightarrow [\delta_B, \delta_{h_Z}] = 0$

$$\circledast = -i \sum_{\substack{Z \subseteq \Lambda \\ Z \cap y = \emptyset}} \delta_{h_Z}(\delta_B(A(\epsilon))) - i \sum_{\substack{Z \subseteq \Lambda \\ Z \cap y \neq \emptyset}} \delta_B \circ \delta_{h_Z}(A(\epsilon))$$

$$H_y = \sum_{Z \subseteq y} h_Z \qquad H_{\Lambda \setminus y} = \sum_{\substack{Z \subseteq \Lambda \\ Z \cap y = \emptyset}} h_Z \qquad H_{\bar{y}} = H_\Lambda - H_{\Lambda \setminus y}$$

$$= \sum_{\substack{Z \subseteq \Lambda \\ Z \cap y \neq \emptyset}}$$

$$\circledast = -i\, \delta_{H_{\Lambda \setminus y}} \circ \delta_B (A(\epsilon))$$

$$- i\, \delta_B \circ \delta_{H_{\bar{y}}} (A(\epsilon))$$

now $F(t) = -\delta_B(A(t))$ so what we have is

$$\frac{d}{dt} F(t) = i\,\delta_{H_{\Lambda_S}}(F(t)) - i\,\delta_B \circ \delta_{H_S}(A(t))$$

Duhamel's formula

linear

inhomogeneous

$$\frac{d}{dt} u(t) = L u(t) + F(t)$$

$$\Rightarrow \boxed{u(t) = e^{tL} u(0) + \int_0^t e^{(t-s)L} F(s)\, ds}$$

Proof: $u(0) = u(0)$ ✓

$$\frac{d}{dt} u(t) = L e^{tL} u(0) + L \int_0^t e^{(t-s)L} F(s)\, ds$$

$$+ e^{tL} \cdot \frac{d}{dt} \int_0^t e^{-sL} F(s)\, ds$$

$$= L u(t) + \cancel{e^{tL}}\, \cancel{e^{-tL}}\, F(t) = L u(t) + F(t) \quad \square$$

Duhamel

$$\circledast \overset{?}{=} \qquad L = i\delta_{H \wedge y} \qquad F = -i\delta_B \circ \delta_{H_{\bar{y}}}(A(t))$$

$$e^{tL} = \alpha_t^{\wedge y}$$

$$\Rightarrow \quad f(t) = \alpha_t^{\wedge y}(f(0)) - i \int_0^t \alpha_{t-s}^{\wedge y} \circ \delta_B \circ \delta_{H_{\bar{y}}}(A(s))\, ds$$

<u>Take norms</u>

$$\| f(t) \| = \| [A(t), B] \|$$

$$\leq \| f(0) \| + \int_0^t 2\|B\| \cdot \| [A(s), H_{\bar{y}}] \|\, ds$$

divide by $2\|B\|$ and take a sup

$$C_A(y;t) \leq C_A(y;0) + \sum_{Z \cap y \neq \emptyset} \|h_Z\| \cdot \int_0^t C_A(Z;s)\, ds$$

Now let $J = \sup_Z \|h_Z\| < +\infty$ (finite range)

$$C_A(y;t) \leq C_A(y;0) + J \sum_{z \cap y \neq \emptyset} \int_0^t C_A(z;s)\,ds$$

$$\leq C_A(y;0) +$$

$$Jt \sum_{z \cap y \neq \emptyset} C_A(z;0) +$$

$$\frac{J^2 t^2}{2} \sum_{\substack{z_1 \cap y \neq \emptyset \\ z_2 \cap z_1 \neq \emptyset}} C_A(z_2;0) + \cdots$$

$$= \sum_{n=0}^{N} \frac{(Jt)^n}{n!} a_n + R_N(t)$$

$$R_N(t) = \sum_{z_1 \cap y \neq \emptyset} \sum_{z_2 \cap z_1 \neq \emptyset} \cdots \sum_{z_N \cap z_{N-1} \neq \emptyset} \int_0^t dt_1 \cdots \int_0^{t_{N-1}} dt_N \; C_A(z;s)\,ds$$

$$\leq \|A\| \cdot \frac{t^N}{N!} \cdot \sum_{z_1 \cap y \neq \emptyset} \cdots \sum_{z_N \cap z_{N-1} \neq \emptyset} 1$$

$$\leq \|A\| \frac{t^N}{N!} \sum_{y \in Y} \sum_{|y_1 - y| \leq r_0} \cdots \sum_{|y_N - y_{N-1}| \leq r_0} 1$$

$$= \|A\| \frac{t^N}{N!} \cdot |Y| \cdot (r_0^{\,V})^N \longrightarrow 0 \quad \text{as } N \to +\infty$$

$$\Rightarrow \quad C_A(y;t) = \sum_{n \geq 0} a_n \frac{(Jt)^n}{n!}$$

$$a_n = \sum_{Z_1 \cap y \neq \emptyset} \; \sum_{Z_2 \cap Z_1 \neq \emptyset} \cdots\cdots \sum_{Z_n \cap Z_{n-1} \neq \emptyset} C_A(Z_n, 0)$$

$$a_1 \leq \sum_{y_0 \in Y} \; \sum_{|y_1 - y_0| \leq r_0} \|A\| = |y| \, r_0^{\nu} \cdot \|A\|$$

$$a_2 \leq \sum_{y_0 \in J} \; \sum_{|y_1 - y_0| \leq r_0} \; \sum_{|y_2 - y_1| \leq r_0} \|A\| = \|A\| \cdot |y| \cdot r_0^{2\nu}$$

$$\vdots$$

$$a_n \leq \|A\| \cdot |y| \; r_0^{n \cdot \nu}$$

$$\Rightarrow \quad C_A(y;t) \leq \sum_{n \geq 0} \frac{(J \cdot r_0^{\nu} \cdot t)^n}{n!} \|A\| = \|A\| \, e^{J r_0^{\nu} t}$$

but $a_n = 0$ if $n < \frac{d}{r_0}$

$$\sum_{n \geq k} a_n \frac{x^n}{n!} = \frac{x^k}{k!} \cdot \sum_{n \geq 0} a_{n-k} \frac{x^n}{n!}$$

Let $$S = \sup_i \sum_{\substack{z \ni i \\ \mathrm{diam}\, z \le r_0}} |z| \, e^{\mu\, \mathrm{diam}\, z} < +\infty$$

we have that

$$C_A(y;t) = \sum_{n \geq 0} a_n \frac{(J|t|)^n}{n!}$$

and $$a_n = \sum_{z_1 \cap y \neq \emptyset} \sum_{z_2 \cap z_1 \neq \emptyset} \cdots \cdots \sum_{z_n \cap z_{n-1} \neq \emptyset} C_A(z_n, 0)$$

Claim $$a_n \leq \sum_{y_0 \in y} e^{-\mu\, \mathrm{dist}(y_0, X)} S^n \|A\|$$

y_1 needs to be in X otherwise this is 0!

$$a_1 \leq \|A\| \cdot \sum_{y_0 \in y} \sum_{z_1 \ni y_0} \sum_{y_1 \in z_1} 1$$

$$\leq \|A\| \cdot \sum_{y_0 \in y} e^{-\mu\, \mathrm{dist}(y_0, X)} \sum_{z_2 \ni y_0} \sum_{y_1 \in z_1} e^{\mu\, \mathrm{dist}(y_0, y_1)}$$

$e^{\mu\, \mathrm{dist}}$

$$\leq \|A\| \cdot \sum_{y_0 \in y} e^{-\mu\, \mathrm{dist}(y_0, X)} \sum_{z_2 \ni y_0} |z_1| \, e^{\mu\, \mathrm{diam}\, z_1}$$

$$\leq \|A\| \cdot \sum_{y_0 \in} e^{-\mu\, \mathrm{dist}(y_0, X)} \cdot S$$

$$Q_2 \le \|A\| \sum_{y_0 \in Y} \sum_{z_1 \ni y_0} \sum_{y_1 \in z_1} \sum_{z_2 \ni y_1} \sum_{y_2 \in z_2 \cap X} 1$$

$$\le \|A\| \sum_{y_0 \in Y} \sum_{z_1 \ni y_0} \sum_{y_1 \in z_1} e^{-n\,\mathrm{dist}(y_1, x)} \underbrace{\sum_{z_2 \ni y_1} \sum_{y_2 \in z_2 \cap X} e^{n\,\mathrm{dist}(y_1,}}_{S}$$