# Lecture 5 - 05/12/2022

Sunday, 4 December 2022 21:21

Lieb-Robinson bounds imply localized approximations of observables

$$\alpha_t^{\Lambda_1}(X) = e^{itH_{\Lambda_1}} X e^{-itH_{\Lambda_1}}$$

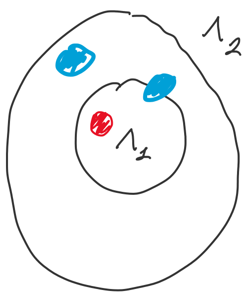
$$f(t) := \alpha_t^{\Lambda_1}(X) - \alpha_t^{\Lambda_2}(X)$$

$$f(0) = X - X = 0$$

$$\frac{d}{dt} f(t) = i\left[H_{\Lambda_1}, \alpha_t^{\Lambda_1}(X)\right] - i\left[H_{\Lambda_2}, \alpha_t^{\Lambda_2}(X)\right]$$

$$= i\left[H_{\Lambda_1}, \alpha_t^{\Lambda_1}(X) - \alpha_t^{\Lambda_2}(X)\right]$$

$$+ i\Big[\underbrace{H_{\Lambda_1} - H_{\Lambda_2}}_{=\sum_{\substack{Z \subseteq \Lambda_2 \\ Z \not\subseteq \Lambda_1}} h_Z}, \alpha_t^{\Lambda_2}(X)\Big]$$




= included • = excluded

$$= i\left[H_{\Lambda_1}, f(t)\right] + i\left[H_{\Lambda_1} - H_{\Lambda_2}, \alpha_t^{\Lambda_2}(X)\right]$$

Duhamel's formula $\Rightarrow$

$$\alpha_t^{\Lambda_1}(X) - \alpha_t^{\Lambda_2}(X)$$

$$= \underset{f(0)=0}{0} + \int_0^t \alpha_{t-s}^{\Lambda_1}\left(i\left[H_{\Lambda_1} - H_{\Lambda_2}, \alpha_s^{\Lambda_2}(X)\right]\right) ds$$

$$\left\| \alpha_t^{\Lambda_1}(X) - \alpha_t^{\Lambda_2}(X) \right\| \leq \int_0^t \sum_{\substack{Z \subseteq \Lambda_2 \\ Z \not\subseteq \Lambda_1}} \left\| \left[h_Z, \alpha_s^{\Lambda_2}(X)\right] \right\| ds$$

$$\leq J\|X\| \cdot \sum_{\substack{Z \subseteq \Lambda_2 \\ Z \not\subseteq \Lambda_1}} |Z| \, e^{-\mu \, dist(X, Z)} \int_0^t e^{\mu v t}$$

$$\leq J\|X\| (r_0)^D \frac{e^{\mu v t} - 1}{\mu v} \sum_{\substack{Z \subseteq \Lambda_2 \\ Z \not\subseteq \Lambda_1}} e^{-\mu \, dist(X, Z)} \longrightarrow 0$$

as $\Lambda_1, \Lambda_2 \nearrow \Gamma$

$\Rightarrow$ The sequence is Cauchy and we can define

$$\alpha_t^{\Gamma}(X) = \lim_{\Lambda \to \Gamma} \alpha_t^{\Lambda}(X)$$

Quite interestingly: Let $\varepsilon > 0$. How big do we need $\Lambda_1$ to be in order to have

$$\| \alpha_t^{\Lambda_2}(X) - \alpha_t^{\Lambda_1}(X) \| \leq \varepsilon \|X\| \quad \forall \Lambda_2 \supseteq \Lambda_1 \; ?$$

$$J (r_0)^D \frac{e^{\mu v t} - 1}{\mu v} \sum_{\substack{z \in \Lambda_2 \\ z \notin \Lambda_1}} e^{-\mu \, dist(x, z)}$$

$$\leq J (r_0)^D \frac{e^{\mu v t} - 1}{\mu v} e^{-\mu [dist(X, \Lambda_1^c) - r_0]} \overbrace{\sum_{\substack{z \in \Gamma \\ z \notin \Lambda_1}} e^{-\mu \, dist(\Lambda_1^c, z)}}^{\text{bounded}}$$

$\leq \varepsilon$ if we chose

$$dist(X, \Lambda_1^c) \geq c \cdot v t$$

for some $c$ depending on $r_0$, $D$, $\Gamma$.

Linear growth of the support of a good approximation of the evolution of $X$!

$$\tilde{X}(t) = \alpha_t^{\Lambda_1(t)}(X)$$

where $\Lambda_1(t)$ is choosen as above.

The $\quad |\text{Supp} \, \tilde{X}(t)| \approx (c \cdot v \cdot t)^D$

while $\| X(t) - \tilde{X}(t) \| \leq \varepsilon \quad \forall t \geq 0.$