Lecture 6 - 07/12/2022

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Decay of correlations

Let $\omega : \mathcal{A}_\Lambda \to \mathbb{C}$ a state.

Let X, Y two regions $X, Y \subseteq \Lambda$ $X \cap Y = \emptyset$

ρ_{XY} reduced density matrix on $X \cup Y$

(sufficient to describe expectation values of all operators on $X \cup Y$)

We want to describe how correlated regions X and Y under ω.

Covariance:

$$C(X:Y) = \sup_{\substack{A \in \mathcal{A}_X \\ B \in \mathcal{A}_Y \\ \|A\| \leq 1 \\ \|B\| \leq 1}} | \omega(A \otimes B) - \omega(A)\,\omega(B) |$$

$$= \sup_{\dots} | Tr(\rho_{XY}(A \otimes B)) - Tr(\rho_X A)\,Tr(\rho_Y B) |$$

$$= \sup_{\dots} | Tr(\rho_{XY}(A \otimes B)) - Tr((\rho_X \otimes \rho_Y)(A \otimes B)) |$$

$$= \sup_{\dots} | Tr((\rho_{XY} - \rho_X \otimes \rho_Y)(A \otimes B)) |$$

It is a distance between ρ_{XY} and $\rho_X \otimes \rho_Y$

if $\rho_{XY} = \rho_X \otimes \rho_Y \Rightarrow$ perfectly uncorrelated

if $\rho_{XY} \neq \rho_X \otimes \rho_Y \Rightarrow$ some correlations are

present and $C(X:Y)$
is a way of quantifying them

Trace distance:

The sup in $C(X:Y)$ is over all
operators $M \in \mathcal{A}_X \otimes \mathcal{A}_Y$ with $\|M\| \leq 1$
and the extra restriction of being of the
form $M = A \otimes B$. (simple tensors)

we can lift the latter restriction:

$$T(A:B) = \sup_{\substack{M \in \mathcal{A}_X \otimes \mathcal{A}_Y \\ \|M\| \leq 1}} \left| \mathrm{Tr}\left((\rho_{XY} - \rho_X \otimes \rho_Y) M \right) \right|$$

$$= \| \rho_{XY} - \rho_X \otimes \rho_Y \|_1$$

$\|\cdot\|_1$ is the trace norm / Schatten-1 norm
(sum of singular values)

From the definition, it is clear that

$$C(X:Y) \leq T(X:Y)$$

The problem is, operationally (e.g. in an actual
experiment), that in order to estimate $T(X:Y)$
one needs to perform "non-local measurements"
i.e. perform joint measurements in space separated
regions. This might be feasible, but it might not be ....

One final comment:

Mutual information:

$$I(X:Y) = S(\rho_X) + S(\rho_Y) - S(\rho_{XY})$$

$S(\rho) = -\mathrm{Tr}\,\rho\log\rho$ is the Von Neumann entropy.

Relationships between trace distance and mutual information:

$$\frac{1}{4}T(X:Y)^2 \overset{(1)}{\leq} I(X:Y) \overset{(2)}{\leq} 6\,T(X:Y)\log\min(\dim \mathcal{H}_X, \dim \mathcal{H}_Y) + 4\,H(T(X:Y))$$

when $H(p) = -p\log p - (1-p)\log(1-p)$ binary entropy

1. Pinsker's inequality
2. Fannes-Audenaert + Alicki-Fannes inequalities

Correlation decay and critical systems

Exponential decay of correlations:

$$C(X:Y) \leq c(|X|,|Y|)\cdot e^{-\frac{1}{\xi}\mathrm{dist}(X,Y)}$$

ξ is known as correlation lenght.

Note: if Λ is finite, this is always trivially true.

- ω is defined ...
- c and ξ are uniform in Λ as $\Lambda \nearrow \Gamma$

Classically, phase transitions have diverging correlation lenghts.

$\Rightarrow$ we expect finite correlation lenght only on "non-critical" states.

Consider

(*) $\omega(A_0 \otimes B_0) - \omega(A_0)\,\omega(B_0)$

For some $A_0 \in \mathcal{A}_x$, $B_0 \in \mathcal{A}_y$.

$$A = A_0 - \omega(A_0)\mathbb{1} \Rightarrow \omega(A) = 0.$$
$$B = B_0 - \omega(B_0)\mathbb{1} \Rightarrow \omega(B) = 0$$

$$A \otimes B = A_0 \otimes B_0 - \omega(A_0)\, B_0 - \omega(B_0)\, A_0 + \omega(A_0)\,\omega(B_0)\,\mathbb{1}$$

$$\omega(A \otimes B) = \omega(A_0 \otimes B_0) - \omega(A_0)\,\omega(B_0) = (*)$$

Thus bounding (*) is the same as bounding $\omega(A \otimes B)$ for operators with zero expectation value

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Decay of correlations for unique, gapped, ground states

Let $H_\Lambda = \sum \lambda \cdot P_\lambda$ spectral decomp.

Rescale H_Λ to have $\min \sigma(H_\Lambda) = 0$.

and let

$$\gamma_\Lambda = \min \sigma(H_\Lambda) \setminus \{0\}$$

spectral gap

For a finite Λ, $\gamma_\Lambda > 0$. When $\inf_{\Lambda \nearrow \Gamma} \gamma_\Lambda \geq \gamma_\Gamma > 0$

we say that the model is gapped

(informally, that it "has" a spectral gap)

<u>THM</u> exponential clustering.

Under the assumptions above

$$|\langle \psi_0 | A \otimes B | \psi_0 \rangle - \langle \psi_0 | A | \psi_0 \rangle \langle \psi_0 | B | \psi_0 \rangle |$$

$$\leq C(A,B)\, e^{-\frac{\mathrm{dist}(x,y)}{\xi}}$$

where $\xi = \frac{c}{\gamma}$ and $C(A,B) \leq c' \|A\| \cdot \|B\| \min(|X|, |Y|)$

<u>Proof:</u> (sketch)

For a given B, such that $\langle \psi_0, B \psi_0 \rangle = 0$ $(*)$ let us define

$$B^+ = \sum_{\lambda, \lambda' \in \sigma(H)} P_\lambda B P_{\lambda'} \, \Theta(\lambda = \lambda')$$

$$\Big|\ 0 \qquad \text{if } t \leq 0$$

Then $B|\psi_0\rangle = \sum_{\lambda,\lambda'} P_\lambda B P_{\lambda'} |\psi_0\rangle$

$$= \underbrace{P_0 B P_0}_{=0} + \sum_{\lambda>0} P_\lambda B |\psi_0\rangle$$

(*)

$$= B^+ |\psi_0\rangle$$

while $(B^+)^* |\psi_0\rangle = \sum_{\lambda,\lambda'} \theta(\lambda-\lambda') P_{\lambda'} B^* P_\lambda |\psi_0\rangle$

$$= \sum_{\lambda'} \theta(-\lambda') P_{\lambda'} B^* P_0$$

$$= P_0 B^* P_0 = 0$$

This implies that

$$\langle \psi_0 | A \cdot B |\psi_0\rangle = \langle \psi_0 | A B^+ |\psi_0\rangle$$

$$= \langle \psi_0 | [A, B^+] |\psi_0\rangle$$

If we could build a good "localized" version of B^+, we would prove the thm.

$$F_\mu(B) = \int_{\mathbb{R}} \mu(dt)\, \hat{\alpha}_t(B) = \int_{\mathbb{R}} \mu(dt)\, e^{itH_1} B e^{-itH_1}$$

In terms of the spectral projections, we can see that

$$F_\mu(B) = \int_{\mathbb{R}} \mu(dt) \sum_{\lambda\lambda'} P_\lambda e^{itH_1} B e^{-itH_1} P_{\lambda'}$$

$$= \sum_{\lambda\lambda'} P_\lambda B P_{\lambda'} \int_{\mathbb{R}} \mu(dt)\, e^{it(\lambda-\lambda')}$$

$$= \sum_{\lambda\lambda'} \check{\mu}(\lambda-\lambda')\, P_\lambda B P_{\lambda'}$$

Now, if we had a μ such that $\check{\mu} = \Theta$, we could write

$$F_\mu(B) = B^+.$$

Unfortunately this is not possible, but it can be fixed by a limiting argument

$$B^+ = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\hat{\alpha}_t(B)}{t - i\varepsilon}\, dt$$

$$B_q = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{-\frac{\cdots}{2q}}}{t - i\varepsilon} \alpha_t(B)\, dt$$

Claim:

$$\| B_q^+ |\psi_0\rangle - B^+ |\psi_0\rangle \| \leq c\, e^{-\frac{q}{2}} \|B\|$$

in fact

$$\langle \psi_\lambda | B_q^+ - B^+ | \psi_0 \rangle = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \underbrace{\int_{\mathbb{R}} \frac{e^{it\lambda}\left[e^{-\frac{(\gamma t)^2}{2q}} - 1 \right]}{t - i\varepsilon}}\, dt$$

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{it\lambda} \frac{e^{-\frac{(\gamma t)^2}{2q}}}{it - \varepsilon}\, dt = \mathcal{F}\left(e^{-\frac{(\gamma t)^2}{2q}} \right) * \mathcal{F}\left(\frac{1}{it - \varepsilon} \right)$$

$$= \frac{1}{\gamma} \sqrt{\frac{q}{2\pi}} \cdot e^{-\frac{(\gamma \omega)^2}{2q}} * \Theta_\varepsilon$$

$$\Theta_\varepsilon(\omega) = \begin{cases} e^{-\varepsilon\omega} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\varepsilon \to 0$

$$= \frac{1}{\gamma} \sqrt{\frac{q}{2\pi}}\, e^{-\frac{(\gamma \omega)^2}{2q}} * \Theta$$

Θ

Why is B_φ^x localized?

Short times $\to$ $\alpha^\Lambda_t(B)$ is localized

long times $\to$ Gaussian suppresses.

$$\int \mu(dt)\, \alpha^\Lambda_t(B) = \underbrace{\int_{-T_0}^{T_0} \mu(dt)\, \alpha^\Lambda_t(B)} + \underbrace{\int_{|t|>T_0} \mu(dt)\, \alpha^\Lambda_t(B)}$$

Can be localized in a ball around B of radius $\sim T_0$

$\downarrow$

Will commute with A!

(if T_0 is small enough)

$\downarrow$

is smaller than

$\|B\| \cdot \mu(\{|t|>T_0\})$