Lecture 7 - 12/12/2022

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SPECTRAL GAP AND PHASE CLASSIFICATION

Remember the definition of spectral gap

$$\gamma_\Lambda = \min \sigma(H_\Lambda) \setminus \{E_0\}$$

$$E_0 = \min \sigma(H_\Lambda)$$

A model is gapped if $\gamma_\Lambda \geq \gamma^* > 0$ uniformly in Λ.

(Gapped) Quantum phases:

Let $\{h_z^{(0)}\}_{z \in \Gamma}$ and $\{h_z^{(1)}\}_{z \in \Gamma}$ two distinct local interactions on the same Γ (with same local dimension of the Hilbert spaces)

An such that

$$\inf_\Lambda \{\gamma_\Lambda^{(0)}, \gamma_\Lambda^{(1)}\} \geq \gamma^* > 0 \quad (\text{both are gapped})$$

We say that $\{h_z^{(0)}\}_z$ and $\{h_z^{(1)}\}_z$ are in the same phase if there exists smooth interpolation (Piecewise - C^1)

$$\{h_z^{(s)}\}_{z \in \Gamma} \qquad s \in [0,1]$$

Such that

$$\inf_{s \in [0,1]} \inf_\Lambda \gamma_\Lambda^{(s)} \geq \gamma^* > 0.$$

Obs: γ^* controls the (exponential) decay of correlations diverge.

Q: Why is it a good definition?

It formalizes the physical intuition that "physical properties" in the same phase do not change abruptly.

THM Automorphic equivalence (Bachmann, Michalakis, Nachtergaele, Sims 2011)

Let $\mathcal{S}(s)$ the set of ground states of $\{h_Z^{(s)}\}_Z$

Then
$$\mathcal{S}(s) = \mathcal{S}(0) \circ \beta_s \qquad s \in [0,1] \qquad (*)$$

where β_s is a unitary isomorphism that satisfies a Lieb-Robinson bound

Comments

Smoothness of expectation values of ground states
$$\omega_s(B) = \omega_0(\beta_s(B)) \qquad s \in [0,1]$$
is smooth in s.

At finite volumes $(*)$ means that ground state projections $P(s)$ satisfy
$$P(s) = U(s)\, P(0)\, U(s)^*$$

where $U(s)$ is a unitary which is solution to

$$U(0) = \mathbb{1} \qquad \frac{d}{ds} U(s) = i\, D(s)\, U(s)$$

$$D(s) = \int_{-\infty}^{+\infty} dt\, W_{\gamma^*}(t)\, e^{itH(s)}\, H'(s)\, e^{-itH(s)}$$

where W_{γ^*} is a weight function depending on γ^*.

- $D(s)$ is a spectral filter (for each $s \in [0,1]$) applied to the operator $H'(s)$

Since $H'(s) = \sum_z h'^{(s)}_z$ is a sum of local terms,

Lieb-Robinson bounds + decay of $W_{\gamma^*}(t)$ imply that

$D(s)$ is approximated well by a sum of local terms

"quasi-local Hamiltonian"

$\Rightarrow$ $U(s)$ also satisfies a Lieb-Robinson bound (needs a stronger result than the one we have seen in class)

- Ground states vectors

$$P(s) \ni |\psi_s\rangle = \underbrace{U(s)}_{\text{unitary evolution generated by } D(s)} |\psi_0\rangle \qquad \forall\, |\psi_0\rangle \in P(0)$$

$$\frac{d}{ds} |\Psi_s\rangle = i\, D(s)\, |\Psi_s\rangle$$

Since $D(s)$ is a quasi-local Hamiltonian, $U(s)$ has Lieb-Robinson light cones.

We won't prove this result, but instead focus on

How to prove a spectral gap?

Let us assume for simplicity

- Interactions $\{h_z\}_{z \in \Gamma}$ are finite-range
- they are translation invariant :

  Let Γ be a lattice (e.g. $\mathbb{Z}^D$ for some $D \in \mathbb{N}$)

$$h_{z+v} = h_z \qquad \forall\, v \in \Gamma$$

These two assumptions imply that we only need to define a finite number of parameters.

E.g. Ising model with transverse field

- single site $\qquad \mu\, \sigma^x$
- nearest neighbor $\qquad -J\, \sigma^z \otimes \sigma^z$

$$H = -J \sum_{i \sim j} \sigma_i^z \otimes \sigma_j^z + \mu \sum_i \sigma_i^x$$

( ) tr) t;

$$\inf_{\Lambda \nearrow \Gamma} \gamma_\Lambda \geq \gamma^* > 0$$

For a specific sequence $\Lambda \nearrow \Gamma$?

A: (Cubitt - Perez Garcia - Wolf 2015, 2D
Bausch - Cubitt - Lucia - Perez Garcia 2018, 1D)

The problem is undecidable!

There is no algorithm that can give us an answer in every instance, as the problem is equivalent to asking if a given Turing machine halts!

Nonetheless, this does not mean we can solve the problem for special classes of models!

Frustration free models

A local interaction is Frustration free if, for every Λ, groundstates of H_Λ minimize the energy of each h_Z for $Z \subseteq \Lambda$.

$$gs(H_\Lambda) = \bigcap_{Z \subset \Lambda} gs(H_Z)$$

This is equivalent to requiring

$$h_Z \geq 0 \quad \text{and} \quad \bigcap_{Z \subseteq \Lambda} \ker(h_Z) \neq \{0\} \qquad (F)$$

if (*) holds, then

$$\ker\Big(\sum_{Z\subseteq\Lambda} h_Z\Big) = \bigcap_{Z\subseteq\Lambda} \ker(h_Z)$$

Since

$$\langle\phi | H_\Lambda | \phi\rangle = \langle\phi | \sum_{Z\subseteq\Lambda} h_Z | \phi\rangle = \underbrace{\sum_{Z\subseteq\Lambda} \langle\phi | h_Z | \phi\rangle}_{\text{Sum of positive terms}} \geq 0$$

Moreover, let P_Z projection on Range h_Z

and
$$\tilde{H}_\Lambda = \sum_{Z\subseteq\Lambda} P_Z .$$

Then $\exists\, c_1, c_2 > 0$ such that

$$c_1 \tilde{H}_\Lambda \leq H_\Lambda \leq c_2 \tilde{H}_\Lambda \quad \forall \Lambda .$$

So $\operatorname{gap}(H_\Lambda) \geq c_1 \cdot \operatorname{gap}(\tilde{H}_\Lambda)$

Conclusion: To study (finite range, translation invariant) frustration-free models, it is sufficient to consider local interactions which are projections with non-trivial common kernel

Finite - size criteria

Initiated by Knabe in '88.

OBS H_Λ has gap γ_1 $\Leftrightarrow$ $(H_\Lambda)^2 \geq \gamma_1 H_\Lambda$

$$(H_\Lambda)^2 = \Big(\sum_{Z\subseteq\Lambda} P_Z\Big)^2 = \sum_Z P_Z^2 + \sum_{Z_1\neq Z_2} P_{Z_1}P_{Z_2}$$

$$= H_\Lambda + \sum_{\substack{Z_1\neq Z_2\\ Z_1\cap Z_2\neq\emptyset}} (P_{Z_1}P_{Z_2} + P_{Z_2}P_{Z_1}) + \underbrace{\sum_{\substack{Z_1\neq Z_2\\ Z_1\cap Z_2=\emptyset}} 2P_{Z_1}\cdot P_{Z_2}}_{\geq 0}$$

If P, Q are projections such that $[P,Q]\neq 0$

$\Rightarrow$ $PQ + QP \not\geq 0$!

Lemma

$$PQ + QP \geq -\| PQ - P\wedge Q\| (P+Q)$$

where $P\wedge Q$ = proj range $P\cap$ range Q

Before proving this, let me discuss a consequence

Let $\boxed{\delta = \max_{Z_1,Z_2} \| P_{Z_1}P_{Z_2} - P_{Z_1}\wedge P_{Z_2}\|}$

$P_{Z_1}\wedge P_{Z_2}$ = proj range $P_{Z_1}\cap P_{Z_2}$ = $\big(\text{Proj } GS(Z_1\cup Z_2)\big)^\perp$

OBS 1 $\|PQ - P\wedge Q\| = \|P^\perp Q^\perp - P^\perp \wedge Q^\perp\|$

where $P^\perp = (1-P)$

in fact

$P^\perp Q^\perp = (1-P)(1-Q) = 1 - P - Q + PQ$

$P^\perp \wedge Q^\perp = \operatorname{Proj} \operatorname{range} P^\perp \cap \operatorname{range} Q^\perp$

$= \operatorname{Proj}(\ker P \cap \ker Q) = (1-P) + (1-Q) - (1 - P\wedge Q)$

$= 1 - P - Q + P\wedge Q$