Lecture 8 - 14/12/2022

Tuesday, 13 December 2022 16:51

Finishing last lecture proof:

$$P \wedge Q = \text{proj range } P \cap \text{range } Q$$

$$P \vee Q = \text{proj range } P + \text{range } Q$$

$$PQ + QP = (P + Q - P \vee Q)(P + Q) \geq -c(P+Q)$$

$$\Uparrow$$

$$P + Q \geq (1-c) P \vee Q$$

$$P_0 = P - P \wedge Q \qquad Q_0 = Q - P \wedge Q \implies P_0 = (P \wedge Q)^{\perp} P$$

$$P_0 + Q_0 \geq (1-c) P_0 \vee Q_0$$

$$\| P_0 Q_0 \| = \| PQ - P \wedge Q \|$$

Last lecture argument says that if

For some $0 < \lambda \leq 1$

$$(P_0 + Q_0) |\psi\rangle = (1-\lambda) |\psi\rangle$$

Then $\quad |\psi\rangle = |\psi_p\rangle + |\psi_q\rangle$

and $\quad \langle \psi_p | \psi_q \rangle = -\lambda \|\psi_p\| \|\psi_q\| \leq 0$

So

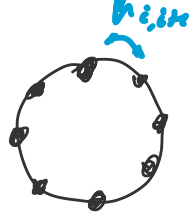
$$\lambda \|\psi_p\| \|\psi_q\| = -\langle \psi_p | \psi_q \rangle$$

$$= |\langle \psi_p | \psi_q \rangle| \leq \| P_0 Q_0 \| \|\psi_p\| \|\psi_q\|$$

$\implies \lambda \leq c$ and $\quad P_0 + Q_0 \geq (1-c) P_0 \vee Q_0$ as we wanted

Going back to finite-size Criteria

$\Gamma = \mathbb{Z}_n$ Periodic boundary Conditions



$$H_n = \sum_{i=0}^{n-1} h_{i,i+1} \qquad h_{i,i+1} \quad \text{orth. Projection}$$

$$(H_n)^2 = \sum_{i=0}^{n} h_{i,i+1} + \sum_{i<j} h_{i,i+1} h_{j,j+1} + h_{j,j+1} h_{i,i+1}$$

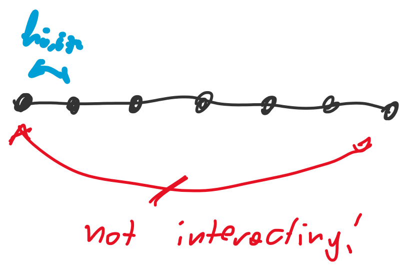
$$\geq H_n - \delta \sum_i \left(h_{i,i+1} + h_{i+1,i+2} \right)$$

$$\geq (1-2\delta) H_n \quad \text{and if } \delta < \tfrac{1}{2} \text{ we have Gap}$$

Knabe 1988

Consider the same model with open boundary Conditions

$$\tilde{H}_K = \sum_{i=0}^{K-1} h_{i,i+1}$$



$K+1$ sites

and suppose we know $\tilde{\gamma}_K = \mathrm{gap}(\tilde{H}_K)$ for some fixed K

So we can say $(\tilde{H}_K)^2 \geq \gamma_K \tilde{H}_K$

$$\Updownarrow$$

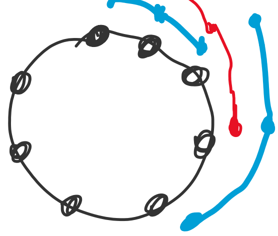
$$\left(\sum_{i=0}^{K-1} h_{i,i+1} \right)^2 \geq \gamma_K \left(\sum_{K=0}^{K-1} h_{i,i+1} \right)$$

we want to compare

$$(H_n)^2 = \left(\sum_{i=0}^{n} h_{i,i+1} \right)^2$$

e.g. $K \sim 3$

$$\sum_{i=0}^{n} \left(\sum_{j=i}^{i+k-1} h_{j,j+2} \right)^2$$



etc.....

$$H_n^2 = H_n + Q + R$$

where $Q = \sum_{i=0}^{n} h_{i,i+2}\, h_{i+1,i+2} + h_{i+2,i+4}\, h_{i+2,i}$

$$R = \sum_{|i-j| \geq 2} h_{i,i+2}\, h_{j,j+2} \geq 0$$

$c \leq k-1$

$$\sum_{i=0}^{n} \left(\sum_{j=1}^{i+k-1} h_{j,j+2} \right)^2 = k H_n + (k-1) Q + c R$$

$$\Rightarrow \frac{1}{k-1} \sum_{i=0}^{n} \left(\sum_{j=1}^{i+k-1} h_{j,j+2} \right)^2 = \left(H_n + Q + \frac{c}{k-1} R \right) + \frac{1}{k-1} H_n$$

$$\leq H_n^2 + \frac{1}{k-1} H_n$$

or $$H_n^2 \geq \frac{1}{k-1} \sum_{i=0}^{n} \left(\sum_{j=1}^{i+k} h_{j,j+2} \right)^2 - \frac{1}{k-1} H_n$$

$$\geq \frac{\tilde{\gamma}_k}{k-1} \sum_{i=0}^{n} \sum_{j=1}^{i+k-1} h_{j,j+2} - \frac{1}{k-1} H_n$$

$$= \left(\frac{k}{k-1} \gamma_k - \frac{1}{k-1} \right) H_n = \frac{k}{k-1} \left(\tilde{\gamma}_k - \frac{1}{k} \right) H_n$$

$$\Rightarrow \gamma_n \geq \frac{k}{k-1} \left(\tilde{\gamma}_k - \frac{1}{k} \right)$$

Finite size

ONLY DEPENDS on $\tilde{\gamma}_n$!

If $\tilde{\gamma}_n < \frac{1}{n}$ for some fixed n $\Rightarrow$ $\gamma_n \geq \gamma^* > 0$
$\forall n$!

Knabe 1988 1D AKLT

(3 sites) $\tilde{\gamma}_2 = 0.5$ vs. $1/2$ no!

(4 sites) $\tilde{\gamma}_3 = 0.4484$ vs. $1/3$ ok! $\longrightarrow$ gap!

OBS if we can show that $\tilde{\gamma}_n \geq \mu_n \to 0$ but slower than $\frac{1}{n}$

Then at some point $\mu_n - \frac{1}{n} > 0$ and $\gamma_n \geq \gamma^* > 0$!

Gosset - Mozgunov 2016

$$\gamma_n \geq c_n \cdot \left(\tilde{\gamma}_n - \frac{6}{n^2} \right)$$

as well as condition in 2D

Lemm 2020 Only D, scaling of $\frac{1}{n}$
Anshu 2020 Only D, Scaling of $\Omega\left(\frac{1}{n^2}\right)$

+ versions for open boundary conditions (with different scaling)

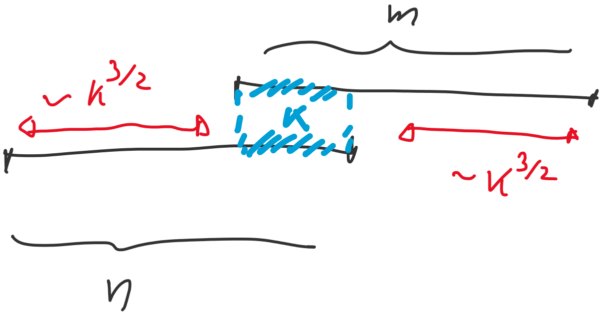
$\Rightarrow$ If a model is gapless $\Rightarrow$ gap has to vanish sufficiently fast w.r.t. system size!

Recursive methods (Lucia - Kastoryano 2017)

L 1 $\|$ P P O $\|$

Ground state projections

i.e. sup over all pair of intervals with intersection of size k



and m, n are at most $\sim k^{3/2}$

THM T.F.A.E.

i) $\tilde{\gamma}_n \geq \gamma^* > 0 \quad \forall n$ open b.c. (gap)

ii) $\delta_k \to 0$ exponentially fast in k

i.e. $\delta_k \equiv \lambda^k$ for some $\lambda < 1$

iii) $\sum_{k \geq 0} \delta_k < +\infty$ (i.e. δ_k is summable)

If one is willing to call δ_k a "STRONG CORRELATIONS MEASURE"

Then THM says

decay of correlations (summable)

gap

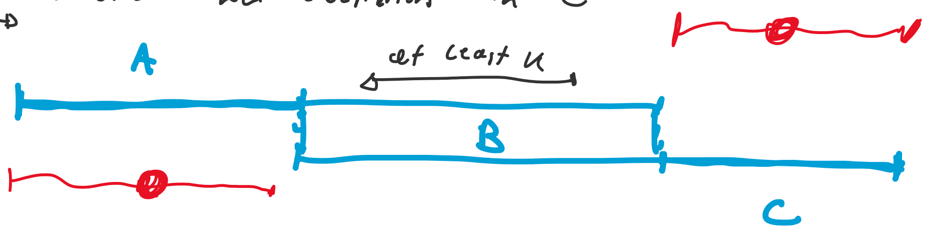
decay of correlations (exp)

Why is δ_k a measure of correlations?

$$\| P_{AB} P_{BC} - P_{ABC} \| = \| (P_{AB} - P_{ABC})(P_{BC} - P_{ABC}) \|$$

$$= \sup_{\substack{|\psi\rangle \in gs(AB) \cap gs(ABC)^{\perp} \\ |\varphi\rangle \in gs(BC) \cap gs(ABC)^{\perp}}} \frac{|\langle \psi | \varphi \rangle|}{\|\psi\| \cdot \|\varphi\|}$$

$|\psi\rangle \rightarrow$ can only have "excitations" in C



$|\varphi\rangle \rightarrow$ can only have "excitations" in A