A simple proof of Lieb-Robinson bounds

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The aim of this version of the proof of Lieb-Robinson bounds is not to be general, but to be pedagogical. The proofs ideas and structure are taken from [1, 2, 3] but I have tried to simplify them at the cost of loosing in generality. Any mistakes are of my own make.

Fix $\Lambda \subset \Gamma$, and let A be an observable $A \in \mathcal{A}_X$, where $X \subset A$ is its support. Let $A(t) = \alpha_t^\Lambda(A)$ the Heisenberg evolution of A according to the Hamiltonian H_Λ of Λ. We will denote

$$C_A(Y;t) := \sup_{B \in \mathcal{A}_Y} \frac{\|[A(t), B]\|}{2\|B\|}. \tag{1}$$

The trivial bound for $t = 0$ is the following:

$$C_A(Y;t) \leq \begin{cases} \|A\| & \text{if } X \cap Y \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Theorem 1 (Lieb-Robinson, 1972)**.** *If H_Λ is a local Hamiltonian with finite-range interaction, then we have that, for all $t \in \mathbb{R}$,*

$$C_A(Y;t) \leq \|A\| \min(|X|, |Y|) e^{\mu(vt - \operatorname{dist}(X,Y))}, \tag{3}$$

where $v = JS_\mu/\mu$ and S_μ is defined in (5), $\mu \geq 0$ is arbitrary and $J = \sum_Z \|h_Z\|$.

We will prove this theorem in two steps.

Lemma 2. *The quantity $C_A(Y;t)$ satisfies, for every $t \in \mathbb{R}$,*

$$C_A(Y;t) \leq C_A(Y;0) + \sum_{Z \cap Y \neq \emptyset} \|h_Z\| \int_0^t C_A(Z;s)\, \mathrm{d}s\,. \tag{4}$$

Proof. Fix $B \in \mathcal{A}_Y$. We start by considering the function $f(t) = [A(t), B]$. We have that $f(0) = [A, B] \leq 2\|B\| C_A(Y;0)$ and that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \left[\frac{\mathrm{d}}{\mathrm{d}t} A(t), B\right] = i[\delta_{H_\Lambda}(A(t)), B] = -i\delta_B \circ \delta_{H_\Lambda}(A(t))$$

where $\delta_W(\cdot) = [W, \cdot]$. One can easily verify (either by direct calculation or via Jacobi's identity) that, if $[W, T] = 0$, then also $[\delta_W, \delta_T] = 0$, in the sense that $[W, [T, \cdot]] = [T, [W, \cdot]]$. This in particular happens when W and T have disjoint support.

Since H_Λ is a sum of local interaction terms $\sum_{Z \subset \Lambda} h_Z$, we can separate the terms whose support is disjoint from Y, which always commute with B, and the terms whose support intersects Y:

$$H_\Lambda = H_{\Lambda \setminus Y} + \big(H_\Lambda - H_{\Lambda \setminus Y}\big).$$

Then also δ_{H_Λ} decomposes similarly

$$\delta_{H_\Lambda} = \delta_{H_{\Lambda \setminus Y}} + \delta_{H_\Lambda - H_{\Lambda \setminus Y}},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = i\delta_{H_{\Lambda\setminus Y}} \circ \delta_B(A(t)) - i\delta_B \circ \delta_{H_\Lambda - H_{\Lambda\setminus Y}}(A(t)) = i\delta_{H_{\Lambda\setminus Y}}(f(t)) + g(t).$$

From Duhamel's formula (6), the solution to this differential equation is given by

$$f(t) = \alpha_t^{\Lambda\setminus Y}(f(0)) + \int_0^t \mathrm{d}s\, \alpha_{t-s}^{\Lambda\setminus Y}(g(s)) = \alpha_t^{\Lambda\setminus Y}([A,B]) - i\int_0^t \mathrm{d}s\, \alpha_{t-s}^{\Lambda\setminus Y} \circ \delta_B \circ \delta_{H_\Lambda - H_{\Lambda\setminus Y}}(A(s)).$$

We now take norms, and using the fact that $\|\delta_W\| \le 2\|W\|$ and that $\|\alpha_t^{\cdot}\| = 1$, we obtain

$$\|f(t)\| \le \|f(0)\| + 2\|B\| \sum_{Z\cap Y\neq\emptyset} \int_0^t \|[h_Z, A(s)]\|\, \mathrm{d}s\,.$$

Dividing by $2\|B\|$ and taking a supremum, we conclude that

$$C_A(Y;t) \le C_A(Y;0) + \sum_{Z\cap Y\neq\emptyset} \|h_Z\| \int_0^t C_A(Z;s)\, \mathrm{d}s\,.$$

□

Note that the proof of the previous lemma does not become any simpler by assuming that h_Z are finite range, and it works as stated for any choice of interaction terms. The next step will require to apply the formula (4) recursively, in which case the decay of $\|h_Z\|$ is crucial. To simplify the construction as much as possible, let us assume that r_0 is the range of the interaction, namely, that h_Z is 0 if $\operatorname{diam}(Z) > r_0$. Similarly, let $J = \sup_Z \|h_Z\|$. We will need to do some counting arguments, which will be simplified by using the following lemma.

Lemma 3. *Let us define, for any $\mu \ge 0$,*

$$S_\mu = \sup_{y\in\Gamma} \sum_{\substack{Z\ni y \\ \operatorname{diam} Z \le r_0}} |Z| e^{\mu \operatorname{diam}(Z)}. \tag{5}$$

Let x, y two points in Γ. Then

$$\sum_{\substack{Z\ni y \\ \operatorname{diam} Z \le r_0}} \sum_{z\in Z} e^{-\mu \operatorname{dist}(x,z)} \le e^{-\mu \operatorname{dist}(y,x)} S_\mu. \tag{6}$$

Proof. Fix $Z \ni y$ and $z \in Z$. Since

$$\operatorname{dist}(x,y) \le \operatorname{dist}(x,z) + \operatorname{dist}(z,y) \le \operatorname{dist}(x,z) + \operatorname{diam} Z,$$

we have that for all $\mu \ge 0$,

$$e^{-\mu \operatorname{dist}(x,z)} \le e^{\mu \operatorname{diam}(Z)} e^{-\mu \operatorname{dist}(y,x)}.$$

Substituting, we obtain

$$\sum_{\substack{Z\ni y \\ \operatorname{diam} Z \le r_0}} \sum_{z\in Z} e^{-\mu \operatorname{dist}(x,z)} \le e^{-\mu \operatorname{dist}(y,x)} \sum_{\substack{Z\ni y \\ \operatorname{diam} Z \le r_0}} |Z| e^{\mu \operatorname{diam}(Z)} = e^{-\mu \operatorname{dist}(y,x)} S_\mu.$$

□

Lemma 4. *With the assumptions above, the function $C_A(Y;t)$ can be bounded as*

$$C_A(Y;t) \le \|A\| \sum_{n\ge 0} \frac{(Jt)^n}{n!} a_n, \tag{7}$$

where the coefficients a_n are given by

$$a_n := a_n(X,Y) = \sum_{Z_1\cap Y\neq\emptyset} \sum_{Z_2\cap Z_1\neq\emptyset} \cdots \sum_{\substack{Z_n\cap Z_{n-1}\neq\emptyset \\ X\cap Z_n\neq\emptyset}} 1, \tag{8}$$

and the sums are over the sets Z_i with diameter smaller or equal to r_0.

Proof. Formally, the bound is obtained by recursively applying (4) into itself an infinite number of times, each time substituting a term $C_A(Z;s)$ which appears in the integral. Here it is useful to recall the formula for the volume of a simplex:

$$\int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n = \frac{t^n}{n!},$$

and the fact that $C_A(Z;0)$ is zero if $Z\cap X=\emptyset$, and otherwise is smaller than $\|A\|$. Thus we really just need to prove that the remainder term is vanishing. This is given, at step n, by

$$R_n = J^n \sum_{Z_1\cap Y\neq\emptyset} \sum_{Z_2\cap Z_1\neq\emptyset} \cdots \sum_{Z_n\cap Z_{n-1}\neq\emptyset} \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \, C_A(Z_n;t_n).$$

As C_A is always bounded by $\|A\|$, we get the estimate

$$0 \le R_n \le \|A\| \frac{(Jt)^n}{n!} \sum_{Z_1\cap Y\neq\emptyset} \sum_{Z_2\cap Z_1\neq\emptyset} \cdots \sum_{Z_n\cap Z_{n-1}\neq\emptyset} 1,$$

where the r.h.s. is similar to the definition of a_n but without the restriction that $X\cap Z_n\neq\emptyset$. We claim that this quantity grows, as a function of n, slower than $n!$, which is enough to show that R_n vanishes as $n\to\infty$.

We will do an overcounting, as we really just care about the scaling in n. Forgetting for a moment about the terms $\|A\|\frac{(Jt)^n}{n!}$, let us focus on the summations. We need to count sequences $(Z_1,\dots,Z_n)$, where each Z_i has diameter at most r_0, such that $Z_i\cap Z_{i+1}\neq\emptyset$ for every $i=1,\dots,n-1$, and moreover such that $Y\cap Z_1\neq\emptyset$. For notational simplicity, let $Z_0=Y$. Given such a sequence $(Z_1,\dots,Z_n)$, we can choose a point $y_i\in Z_i\cap Z_{i+1}$ for $i=0,\dots,n-1$, building a sequence of n points $(y_0,\dots,y_{n-1})$ such that $\mathrm{dist}(y_i,y_{i+1})\le \mathrm{diam}(Z_{i+1})\le r_0$ for every i. With ample overcounting, there are at most $|Z_0|=|Y|$ ways of choosing y_0, and at most $|Z_i|$ ways to choose y_i for $i>0$. We can therefore bound

$$\sum_{Z_1\cap Y\neq\emptyset} \sum_{Z_2\cap Z_1\neq\emptyset} \cdots \sum_{Z_n\cap Z_{n-1}\neq\emptyset} 1 \le \sum_{y_0\in Y} \sum_{Z_1\ni y_0} \sum_{y_1\in Z_1} \sum_{Z_2\ni y_1} \cdots \sum_{Z_n\ni y_{n-1}} 1.$$

We now use Lemma 3, with the choice $\mu=0$, n times. Each time we can replace a sum over sets containing a given element with a constant, since

$$\sum_{Z\ni z} |Z| \le S_0.$$

This gives us, starting from the trivial $1 \leq |Z_n|$,

$$\sum_{y_0 \in Y} \sum_{Z_1 \ni y_0} \sum_{y_1 \in Z_1} \sum_{Z_2 \ni y_1} \cdots \sum_{Z_n \ni y_{n-1}} 1 \leq \sum_{y_0 \in Y} \sum_{Z_1 \ni y_0} \sum_{y_1 \in Z_1} \sum_{Z_2 \ni y_1} \cdots \sum_{Z_n \ni y_{n-1}} |Z_n|$$
$$\leq S_0 \sum_{y_0 \in Y} \sum_{Z_1 \ni y_0} \sum_{y_1 \in Z_1} \sum_{Z_2 \ni y_1} \cdots \sum_{Z_{n-1} \ni y_{n-2}} \sum_{y_{n-1} \in Z_{n-1}} 1$$
$$\leq S_0 \sum_{y_0 \in Y} \sum_{Z_1 \ni y_0} \sum_{y_1 \in Z_1} \sum_{Z_2 \ni y_1} \cdots \sum_{Z_{n-1} \ni y_{n-2}} |Z_{n-1}|$$
$$\leq \cdots \leq \sum_{y_0 \in Y} S_0^n \leq |Y| S_0^n .$$

This shows that $R_n \to 0$ as $n \to \infty$. □

Lemma 5. *For every choice of $\mu \geq 0$, the coefficients $a_n(X,Y)$ are bounded by*

$$a_n(X,Y) \leq S_\mu^n \sum_{y \in Y} e^{-\operatorname{dist}(X,y)} \leq S_\mu^n |Y| e^{-\mu dist(X,Y)}. \tag{9}$$

Proof. The argument is quite similar to the one we made in the proof of Lemma 4 to bound R_n. Once again, for a sequence $(Z_0 = Y, Z_1, \ldots, Z_n, Z_{n+1} = X)$, such that $Z_i \cap Z_{i+1} \neq \emptyset$ for every i, we choose elements $y_i \in Z_i \cap Z_{i+1}$, and write

$$a_n(X,Y) \leq \sum_{y_0 \in Y} \sum_{Z_1 \ni y_0} \sum_{y_1 \in Z_1} \sum_{Z_2 \ni y_1} \cdots \sum_{Z_n \ni y_{n-1}} \sum_{y_n \in Z_n \cap X} 1.$$

We will apply Lemma 3 n times, with the same choice of $\mu > 0$. Each time, we can replace from the expression of a_n a sum over sets containing a fixed element by a constant, since

$$\sum_{Z \ni y} \sum_{z \in Z} e^{-\mu \operatorname{dist}(x,z)} \leq S_\mu e^{-\mu \operatorname{dist}(x,y)}.$$

To start, since $y_n \in X$, we have that $1 = e^{-\mu \operatorname{dist}(x,y_n)}$. This gives us

$$a_n(X,Y) \leq \sum_{y_0 \in Y} \sum_{Z_1 \ni y_0} \sum_{y_1 \in Z_1} \sum_{Z_2 \ni y_1} \cdots \sum_{Z_n \ni y_{n-1}} \sum_{y_n \in Z_n \cap X} e^{-\mu \operatorname{dist}(x,y_n)}$$
$$\leq S_\mu \sum_{y_0 \in Y} \sum_{Z_1 \ni y_0} \sum_{y_1 \in Z_1} \sum_{Z_2 \ni y_1} \ldots e^{-\mu \operatorname{dist}(x,y_{n-1})} \leq \cdots \leq S_\mu^n \sum_{y_0 \in Y} e^{-\mu \operatorname{dist}(x,y_0)}.$$

□

References

[1] M. B. Hastings. *Locality in Quantum Systems*. 2010. arXiv: 1008.5137.

[2] P. Naaijkens. *Quantum Spin Systems on Infinite Lattices*. Springer International Publishing, 2017. DOI: 10.1007/978-3-319-51458-1. arXiv: 1311.2717.

[3] B. Nachtergaele. *Introduction to Quantum Spin Systems*. Lecture notes. URL: https://www.math.ucdavis.edu/~bxn/.

Lemma 6 (Duhamel's formula)**.** *Let $u : \mathbb{R} \to \mathcal{M}_d$ a matrix-valued function satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = Lu(t) + g(t), \tag{10}$$

for some linear $L : \mathcal{M}_d \to \mathcal{M}_d$ and a non-homogeneous term $g : \mathbb{R} \to \mathcal{M}_d$. Then

$$u(t) = e^{tL} u(0) + \int_0^t e^{(t-s)L} g(s) \, \mathrm{d}s \,. \tag{11}$$