The projections lemma

Angelo Lucia

December 13, 2022

The first proof of this result I know of is in [1, Lemma 6.3], but it might have been older.

Definition 1. For P and Q non-zero orthogonal projections on a finite dimensional Hilbert space $\mathcal{H}$, let $P \wedge Q$ the orthogonal projection on $\operatorname{range} P \cap \operatorname{range} Q$, and $P \vee Q$ the orthogonal projection on $\operatorname{range} P + \operatorname{range} Q$.

Note that, by definition

$$P(P \wedge Q) = P \wedge Q = (P \wedge Q)P, \qquad P(P \vee Q) = P = (P \vee Q)P,$$
$$Q(P \wedge Q) = P \wedge Q = (P \wedge Q)Q, \qquad Q(P \vee Q) = Q = (P \vee Q)Q.$$

These facts will be used repeatedly during the proof.

Lemma 1. *With the notation above*

$$PQ + QP \geq -\|PQ - P \wedge Q\|(P + Q). \tag{1}$$

Proof. To prove the result we will do a few simplifications first. Let $c \geq 0$ a positive constant.

- We claim that $P + Q \geq (1 - c)P \vee Q$ implies that $PQ + QP \geq -c(P + Q)$. This can be seen by rewriting
$$PQ + QP = (P + Q - P \vee Q)(P + Q).$$
Now $P + Q$ and $P + Q - P \vee Q$ commute, and thus if $(P + Q - P \vee Q) \geq -c\,(P \vee Q)$, we have that
$$PQ + QP = (P + Q - P \vee Q)(P + Q) \geq -c\,(P \vee Q)(P + Q) = -c\,(P + Q)$$
which is our claim.

- Consider the decomposition of the identity given by $P \wedge Q$ and $(P \wedge Q)^\perp = P^\perp \vee Q^\perp$. Since both $(P + Q)$ and $P \vee Q$ commute with $P \wedge Q$, we have that $P + Q \geq (1 - c)(P \vee Q)$ if and only if
$$(P \wedge Q)(P + Q) \geq (1 - c)(P \wedge Q)(P \vee Q),$$
$$(P \wedge Q)^\perp(P + Q) \geq (1 - c)(P \wedge Q)^\perp(P \vee Q).$$
Note that the first inequality is always satisfied for any $c \geq 0$, since
$$(P \wedge Q)(P + Q) = 2(P \wedge Q) \geq (1 - c)(P \wedge Q) = (1 - c)(P \wedge Q)(P \vee Q),$$
so we only need to check the second one.

- The restriction on $(P \wedge Q)^\perp$ can be written as follows. Let us define
$$P_0 = P - P \wedge Q, \quad Q_0 = Q - P \wedge Q.$$
Then $P_0 \wedge Q_0 = 0$, $P_0 \vee Q_0 = (P \wedge Q)^\perp(P \vee Q)$, $(P \wedge Q)^\perp(P + Q) = P_0 + Q_0$ and
$$P_0 Q_0 = PQ - P \wedge Q.$$

In conclusion, the statement of the Lemma is proven if we can show that

$$P_0 + Q_0 \geq (1 - c)(P_0 \vee Q_0)$$

for $c = \|P_0 Q_0\| = \|PQ - P \wedge Q\|$. Note that, since $P_0 \vee Q_0$ is the projection on the range of $P_0 + Q_0$, what we really need to prove is that any non-zero eigenvalue of $P_0 + Q_0$ is at least $1 - c$. So let $1 - \lambda$ an eigenvalue of $P_0 + Q_0$: we want to show that $\lambda \leq c$. Clearly, if $\lambda \leq 0 \leq c$, we have nothing to prove. Similarly, since $P_0 + Q_0 \geq 0$, any such eigenvalue satisfies $\lambda \leq 1$. Therefore, we can assume that $0 < \lambda \leq 1$.

Now let $|\phi\rangle \in \operatorname{range} P_0 + \operatorname{range} Q_0 = \operatorname{range}(P_0 \vee Q_0)$. Since $P_0 \wedge Q_0 = 0$, $|\phi\rangle$ has a unique decomposition

$$|\phi\rangle = |\phi_p\rangle + |\phi_q\rangle \,, \quad |\phi_p\rangle \in \operatorname{range} P_0, \quad |\phi_q\rangle \in \operatorname{range} Q_0.$$

Let us now require that $|\phi\rangle$ is an eigenvector of $P_0 + Q_0$ with eigenvalue $1 - \lambda$, for some $0 < \lambda \leq 1$.

This implies that

$$(1 - \lambda)\left(|\phi_p\rangle + |\phi_q\rangle\right) = (P_0 + Q_0)(|\phi_p\rangle + |\phi_q\rangle) = |\phi_p\rangle + |\phi_q\rangle + P\,|\phi_q\rangle + Q\,|\phi_p\rangle \,,$$

or equivalently

$$-\lambda\left(|\phi_p\rangle + |\phi_q\rangle\right) = P_0\,|\phi_q\rangle + Q_0\,|\phi_p\rangle \,.$$

Since $P_0\,|\phi_q\rangle \in \operatorname{range} P_0$ and $Q_0\,|\phi_p\rangle \in \operatorname{range} Q_0$, and the decomposition of $-\lambda\,|\phi\rangle$ is unique, this implies

$$P_0\,|\phi_q\rangle = -\lambda\;|\phi_p\rangle \,, \quad Q_0\,|\phi_p\rangle = -\lambda\;|\phi_q\rangle \,.$$

Taking scalar products

$$-\lambda \|\phi_p\|^2 = \langle \phi_p | \phi_q \rangle \,, \quad -\lambda \|\phi_q\|^2 = \langle \phi_q | \phi_p \rangle \,,$$

i.e. $\langle \phi_p | \phi_q \rangle$ is real and negative, and $\|\phi_p\| = \|\phi_q\|$.

This implies that

$$\lambda \|\phi_p\| \|\phi_q\| = -\langle \phi_p | \phi_q \rangle = |\langle \phi_p | \phi_q \rangle| \leq |\langle P_0 \phi_p | Q_0 \phi_q \rangle| \leq \|P_0 Q_0\| \|\phi_p\| \|\phi_q\|,$$

i.e. $\lambda \leq c$ which is what we wanted to prove. □